The Schroeder-Bernstein Theorem

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Statement of the Theorem

Theorem: Given two sets A, B and two injective functions

$$\begin{aligned} f &: A \to B \\ g &: B \to A \end{aligned}$$

there exists a bijective function

$$h : A \to B$$

Proof: This lecture. $\square$

Reminder: *injective* means 1-to-1, *surjective* means onto, and *bijective* means injective and surjective.

Example: Squares and Cubes

- Consider the following example using subsets of $\mathbb{N}$:

$$
\begin{aligned}
A &= \{0, 1, 4, 9, 16, 25, 36, \ldots\} \\
B &= \{0, 1, 8, 27, 64, 125, 216, \ldots\} \\
f &: A \longrightarrow B \\
&: n \longmapsto n^3 \\
g &: B \longrightarrow A \\
&: n \longmapsto n^2
\end{aligned}
$$

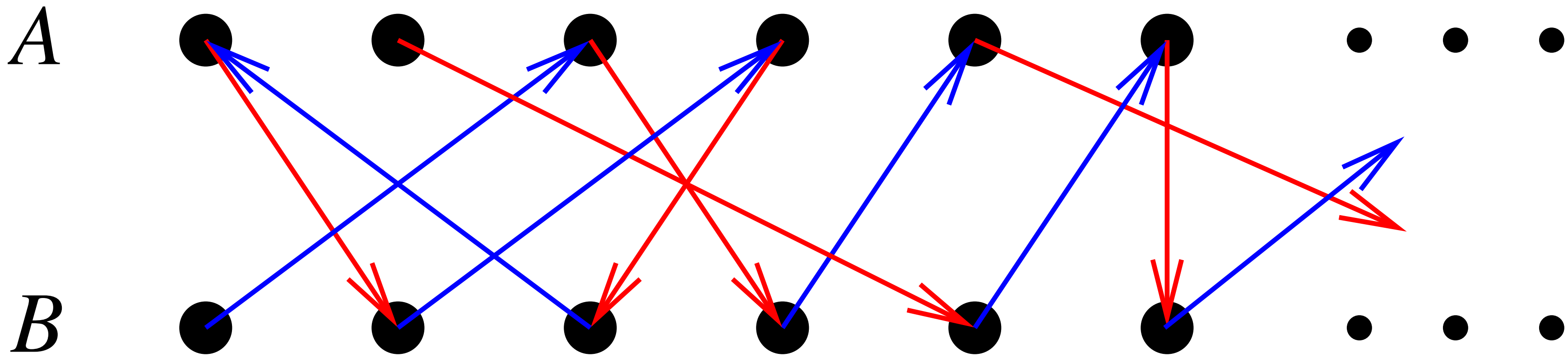
- Need a bijective function $h : A \rightarrow B$.
- Note that f is *not* bijective (it misses out 8).
- One solution is $h : n \mapsto (\sqrt{n})^3$.

How To Prove It?

- Problem: we must prove the theorem without assuming anything about the sets A and B.
- Logically speaking, fewer theorems hold for arbitrary sets than hold for countable sets. *This is obvious!*
- Practically speaking, arbitrary sets have less structure:
  - no handy injection to the natural numbers,
  - so no induction over points.
- For an example of how things can go wrong when sets become uncountable, refer to *The Banach-Tarski Paradox* [a] in probability theory.

[a] [Stan Wagon, Cambridge University Press 1993]

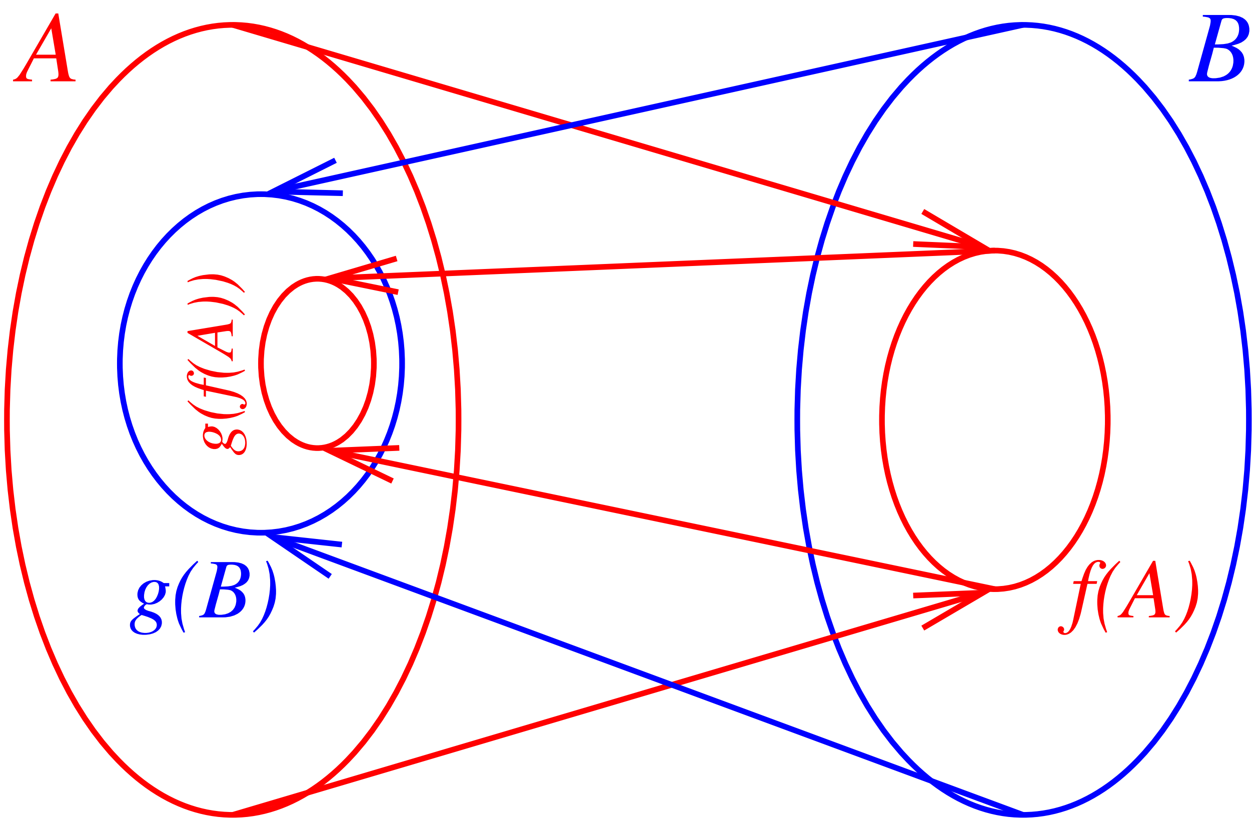
An Informal Proof



- f represented by $\swarrow\downarrow\searrow$, g represented by $\nwarrow\uparrow\nearrow$.
- Every point has 1 arrow out and at most 1 arrow in.
- For any $a \in A$, say that a is B-stopping if making *backward* jumps starting from a ends up in B.
- Define $h(a) = \begin{cases} g^{-1}(a) & \text{if } a \text{ is } B\text{-stopping.} \\ f(a) & \text{otherwise} \end{cases}$

Reducing the Theorem to a Lemma

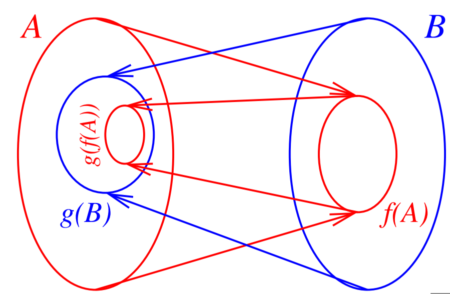
Lemma: Given two sets A, B with $B \subseteq A$, if there exists an injective function $f : A \to B$ there exists a bijective function $h : A \to B$.



Reducing the Theorem to a Lemma

Proof that the lemma implies the theorem:

1. Since f and g are injections, $g \circ f$ is an injection from A to $g(B)$. Also $g(B) \subseteq A$, and the conditions of the lemma are now satisfied, so there must exist a bijection h from A to $g(B)$.
2. It is given that g is an injection from B to A, so g is a bijection from B to $g(B)$. A bijection has an inverse that is a bijection, and so g^{-1} is a bijection from $g(B)$ to B.
3. The composition of two bijections is a bijection, and so $g^{-1} \circ h$ is a bijection from A to B.



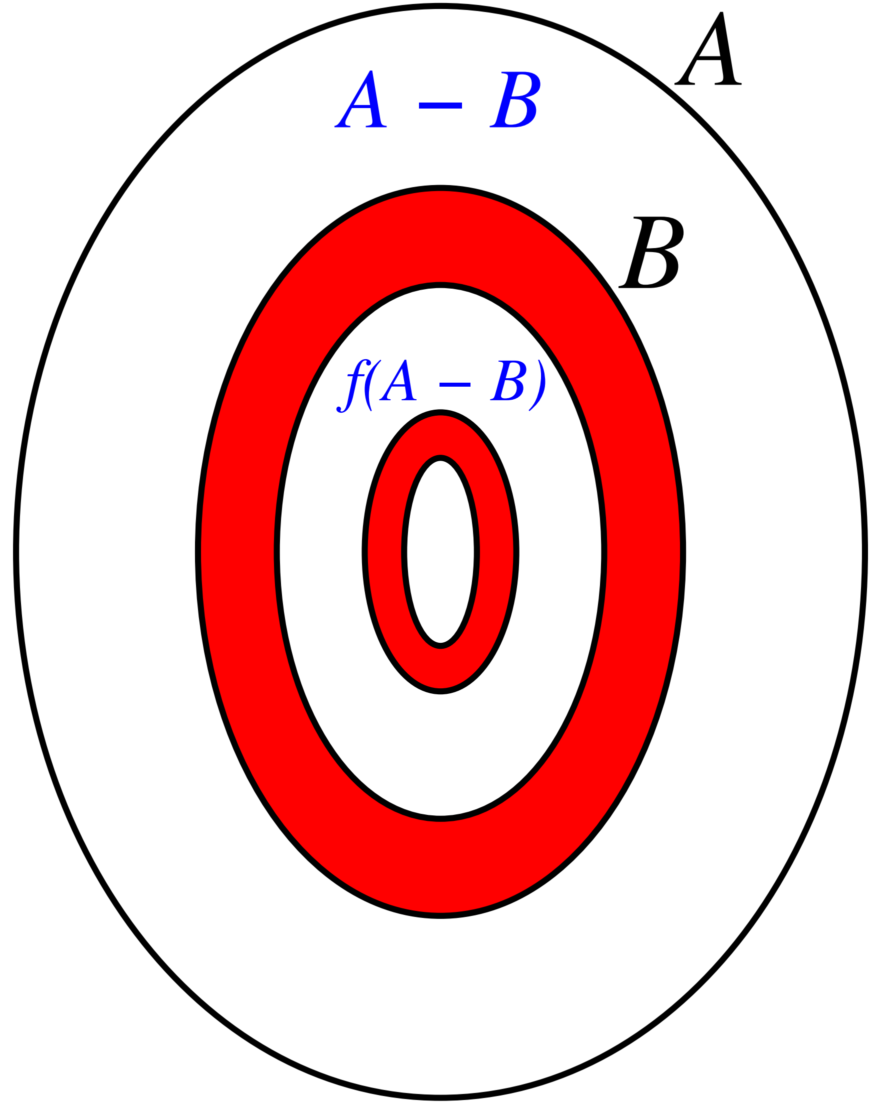
Proving the Lemma

Define

$$X = \bigcup_{0 \le n} f^n(A - B)$$

Claim: The desired bijection is

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \\ x & \text{otherwise} \end{cases}$$

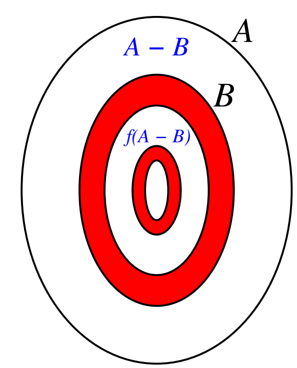


Proving the Lemma

Proof of Claim:

- h **is injective:** Suppose $h(x) = h(y)$. If $x, y \in X$ then $x = y$ since f is injective. If $x, y \notin X$ then $x = y$ by the definition of h. Finally, $f(X) \subset X$ so we cannot have precisely one of x, y in X.
- h **is surjective:** Suppose we have $y \in B$. If $y \in X$, then $y \in f^n(A - B)$ for some $1 \leq n$. Therefore, there exists an $x \in f^{n-1}(A - B) \subseteq X$ satisfying $h(x) = f(x) = y$. If $y \notin X$, then $h(y) = y$ by the definition of h.

We have now proved the Schröder-Bernstein theorem.



Application: 1-to-1 Correspondence

Corollary: The sets A and B are in 1-to-1 correspondence if and only if there exist functions

$$\begin{aligned} f_1 &: A \to B \\ f_2 &: A \to B \end{aligned}$$

where f_1 is injective and f_2 is surjective.

Reminder: Sets A and B are in *1-to-1 correspondence* if there exists a bijective function $h : A \to B$.

Application: 1-to-1 Correspondence

Proof of Corollary:

- If A and B are in 1-to-1 correspondence then there is a bijection $h : A \to B$. Therefore, we can let $f_1 = f_2 = h$.
- Suppose we are given functions f_1, f_2.
  - Define a function $g : B \to A$ by

$$g(y) = \text{an arbitrary } x \text{ such that } f_2(x) = y$$

  - g is well-defined and injective, because f_2 is surjective and a function.
  - Therefore, by Schröder-Bernstein, there exists a bijection $h : A \to B$ and so the sets A and B are in 1-to-1 correspondence.

Application: 1-to-1 Correspondence

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Example: Prove that $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ are in 1-to-1 correspondence.

- Before, we (annoyingly) had to come up with a bijection[a]

$$f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

- Applying the corollary, we (simply) need an injection f_1 and a surjection f_2:

$$\begin{aligned} f_1(k, m, n) &= 2^k 3^m 5^n \\ f_2(k, m, n) &= k \end{aligned}$$

[a] $f(k, m, n) = 2^k(2(2^m(2n + 1) - 1) + 1) - 1$